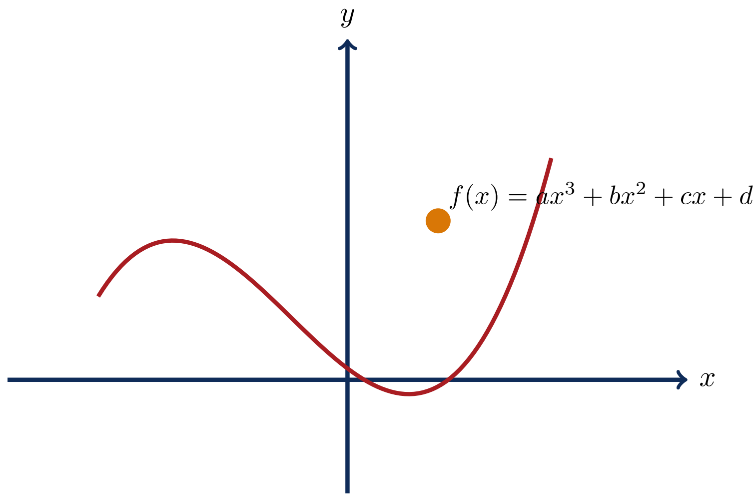
IB MATHEMATICS AI SL

TOPIC 2: FUNCTIONS & MODELLING

15 Long-Response Questions | Graded Medium to Very Hard



Overview & Syllabus Targets

- **Section A: Exam Questions** — 15 structured long-response problems graded in difficulty: Medium (Q1–Q5), Hard (Q6–Q11), and Very Hard (Q12–Q15). Each question begins on a new page.
- **Section B: Worked Solutions** — Step-by-step solutions with GDC parameters, exact marks, and custom Examiner Tips.
- **Syllabus Coverage** — Linear models, quadratic vertices, exponential growth/decay, logarithmic scales, cubic curves, and piecewise continuous optimization models.

SECTION A: EXAM QUESTIONS

Question 1: Linear Modelling & Simultaneous Cost Analysis (Medium) [6 Marks]

Two ride-sharing companies, Company A and Company B, operate in a major metropolitan area. Their fare models are represented as linear functions of the travel distance, x kilometres.

- **Company A** charges an initial booking fee of \$4.50 plus \$1.80 per kilometre.
- **Company B** charges an initial booking fee of \$6.00 plus \$1.50 per kilometre.

1. Write down a linear equation representing the total fare, $F_A(x)$, charged by Company A. **[1 mark]**
2. Write down a linear equation representing the total fare, $F_B(x)$, charged by Company B. **[1 mark]**
3. Find the coordinate intersection point (x, y) of these two linear functions. **[2 marks]**
4. Interpret the physical meaning of this intersection point in the context of journey planning. **[2 marks]**

Question 2: Quadratic Projectile Motion & Maximum Boundaries (Medium) [6 Marks]

A small weather balloon is launched from an elevated research platform. The height, $h(t)$, of the balloon in metres above the ground, t seconds after launch, is modelled by the quadratic function:

$$h(t) = -4.9t^2 + 19.6t + 1.5, \quad t \geq 0$$

1. Write down the initial height of the balloon at the moment of launch. **[1 mark]**
2. Calculate the maximum height reached by the weather balloon, and state the time at which this maximum occurs. **[2 marks]**
3. Calculate the total duration of the balloon's flight before it returns to the ground. **[2 marks]**
4. State the domain and range of the function $h(t)$ in this physical context. **[1 mark]**

Question 3: Exponential Cooling with Asymptotes (Medium) [6 Marks]

The temperature, $T(t)$, of a hot mug of tea in degrees Celsius (°C), t minutes after being poured, is modelled by the exponential decay function:

$$T(t) = 22 + 68e^{-0.08t}, \quad t \geq 0$$

1. Calculate the initial temperature of the tea when it is first poured. **[1 mark]**
2. Find the temperature of the tea after exactly 15 minutes. **[2 marks]**
3. State the equation of the horizontal asymptote of the function $y = T(t)$ and interpret its physical significance in the context of the environment. **[2 marks]**
4. Determine the time it takes for the tea to cool down to a drinkable temperature of 50°C. **[1 mark]**

Question 4: Piecewise Cost Functions for Shipping (Medium) [6 Marks]

An international courier company uses a multi-tier piecewise linear pricing structure to determine the shipping cost, $C(w)$, in dollars, of a parcel of weight w kilograms.

$$C(w) = \begin{cases} 12.00 & 0 < w \leq 2 \\ 12.00 + 2.50(w-2) & 2 < w \leq 10 \\ 32.00 + 1.80(w-10) & w > 10 \end{cases}$$

1. Calculate the cost of shipping a parcel weighing 1.5 kg. **[1 mark]**
2. Calculate the cost of shipping a parcel weighing 5.0 kg. **[2 marks]**
3. Find the weight of a parcel that costs exactly $41.00 to ship. **[2 marks]**
4. Show that the pricing function $C(w)$ is continuous at $w = 10$. **[1 mark]**

Question 5: Inverse Square Law & Light Intensity (Medium) [6 Marks]

The light intensity, $I(d)$, in lux, measured by a sensor placed d metres away from an industrial halogen light bulb, is inversely proportional to the square of the distance from the bulb.
When the sensor is placed exactly 2 m away from the bulb, the measured intensity is 150 lux.

1. Formulate an algebraic equation for $I(d)$ in terms of d. **[2 marks]**
2. Calculate the light intensity when the sensor is moved to a distance of 5 m from the bulb. **[1 mark]**
3. Find the exact distance from the bulb required to obtain an intensity reading of 6 lux. **[2 marks]**
4. Sketch the graph of $y = I(d)$ for $d > 0$, clearly showing the behavior as $d \to 0$ and $d \to \infty$. **[1 mark]**

Question 6: Radioactive Decay & Half-Life Calculations (Hard) [8 Marks]

A diagnostic medical isotope, Technetium-99m, decays exponentially according to the model:

$$N(t) = N_0 e^{-\lambda t}, \quad t \geq 0$$

where $N(t)$ is the mass of the remaining isotope in milligrams, t is the elapsed time in hours, N_0 is the initial mass, and λ is the decay constant.
An initial dose of 200 mg is prepared. The half-life of this isotope is exactly 6.0 hours.

1. Show that the exact value of the decay constant λ is $\frac{\ln(2)}{6}$. **[2 marks]**
2. Calculate the mass of the isotope remaining in the dose after exactly 15 hours. **[2 marks]**
3. Calculate the time required for the mass of the isotope in the dose to decay to exactly 15 mg. **[3 marks]**
4. Write down the mass of the isotope that has decayed away over the first 24 hours. **[1 mark]**

Question 7: Cubic Polynomial Modelling of Water Reservoirs (Hard) [9 Marks]

The volume of water in a local community reservoir, $V(t)$, in cubic decametres (dam^3), over a 10-week observation cycle is modelled by the cubic function:

$$V(t) = -t^3 + 12t^2 - 36t + 200, \quad 0 \leq t \leq 10$$

where t is the time elapsed in weeks.

1. Find the initial volume of water in the reservoir at the start of the observation cycle. **[1 mark]**
2. Find the derivative of the volume function, $V'(t)$. **[2 marks]**
3. Use your derivative to determine the exact coordinates (t, V) of the local minimum and the local maximum points of the curve. **[3 marks]**
4. State the interval of weeks during which the volume of water in the reservoir is strictly increasing. **[1 mark]**
5. Calculate the minimum volume of water recorded in the reservoir during the entire 10-week cycle. **[2 marks]**

Question 8: Sinusoidal Ferris Wheel Mechanics (Hard) [8 Marks]

A giant Ferris wheel with a diameter of 80 m is constructed such that its lowest point is exactly 5 m above the flat ground. The wheel completes one full rotation every 12 minutes.
The height, $H(t)$, in metres of a passenger cabin above the ground t minutes after boarding at the bottom platform is modelled by the trigonometric function:

$$H(t) = -a\cos(bt) + d, \quad t \geq 0$$

where b is measured in radians.

1. Write down the values of the parameters a and d. **[2 marks]**
2. Find the value of b, giving your answer in terms of π. **[2 marks]**
3. Calculate the height of the cabin exactly 4 minutes after the ride begins. **[2 marks]**
4. Calculate the total duration of time, in minutes, that a passenger cabin is strictly above 65 m during the first rotation of the wheel. **[2 marks]**

Question 9: Rational Average Cost vs. Linear Cost Models (Hard) [8 Marks]

A high-capacity digital printing press estimates that the total daily production cost, $C(x)$, in dollars, to print x thousand pages is modelled by the function:

$$C(x) = 1500 + 3.5x, \quad x \geq 0$$

1. Write down an algebraic expression representing the average cost per thousand pages, $A(x) = \frac{C(x)}{x}$. **[1 mark]**
2. Find the vertical asymptote of the average cost function, $A(x)$, and interpret its economic meaning. **[2 marks]**
3. Find the horizontal asymptote of $A(x)$, and describe what it represents as production increases indefinitely. **[2 marks]**

A competing printing firm offers a flat linear rate represented by the cost function:

$$S(x) = 4.20x, \quad x \geq 0$$

4. Calculate the minimum number of thousands of pages that must be printed daily for the high-capacity press's average cost to be strictly cheaper than the competitor's flat rate. **[3 marks]**

Question 10: Multi-Variable Fitting for Quadratic Parabolas (Hard) [8 Marks]

A civil engineer is modeling the shape of a parabolic supporting cable on a suspension bridge. The cable is anchored to three surveyed markers with coordinates:

$$P_1(2, 10), \quad P_2(4, 30), \quad P_3(6, 62)$$

The shape of the cable is modeled by the quadratic function:

$$y = ax^2 + bx + c$$

1. Formulate a system of three linear equations in terms of the parameters a, b, and c. **[3 marks]**
2. Write down the matrix equation representing this system of equations. **[2 marks]**
3. Using your GDC, solve the system of equations to find the exact values of a, b, and c. **[2 marks]**
4. Find the vertical height of the cable at a horizontal distance of $x = 8$. **[1 mark]**

Question 11: Power Models & Logarithmic Scale Linearisation (Hard) [9 Marks]

Biologists use Kleiber's power law, $E = aM^b$, to relate the basal metabolic rate, E (kcal/day), of a mammal to its body mass, M (kilograms), where a and b are constant parameters.

1. Show that the relationship can be linearised into the logarithmic equation:

$$\ln(E) = b\ln(M) + \ln(a)$$

[2 marks]

A set of laboratory measurements yields the following linearised data pairs $(\ln(M), \ln(E))$:

$$(2.77, 6.33), \quad (4.39, 7.54)$$

2. Calculate the value of the parameter b, correct to 2 decimal places. **[3 marks]**
3. Calculate the value of the parameter a, correct to the nearest integer. **[2 marks]**
4. Using the developed model, estimate the metabolic rate of a mammal weighing exactly 16 kg. **[2 marks]**

Question 12: Piecewise Continuous Cost Scale Optimization (Very Hard) [9 Marks]

A chemical synthesis plant operates with a manufacturing cost function, $C(x)$, in dollars, where x represents the mass of chemical output in kilograms. To model the transition from pilot-stage scale-up to high-volume mass production, the firm uses a piecewise function:

$$C(x) = \begin{cases} -0.05x^2 + 8.00x + 50.00 & 0 \leq x \leq 60 \\ mx + c & x > 60 \end{cases}$$

where m and c are real constant parameters.

1. To ensure budget estimates are predictable, the total cost function $C(x)$ must be **continuous** at the boundary $x = 60$. Formulate a linear equation in terms of m and c satisfying this condition. **[2 marks]**
2. To ensure production is smooth, the marginal cost (the rate of change of the cost function) must be **differentiable** at the boundary $x = 60$. Show that the value of m must be exactly 2.00. **[3 marks]**
3. Hence, find the value of the constant c. **[2 marks]**
4. Calculate the total cost to synthesize 100 kg of the chemical under this continuous model. **[2 marks]**

Question 13: Competing Growth Dynamics & Intersection Solvers (Very Hard) [9 Marks]

Two streaming platforms are competing for active monthly users over a 20-month timeline. Let the number of users (in thousands) on Platform A, t months after tracking begins, be modeled by the exponential growth function:

$$A(t) = 250(1.15)^t, \quad 0 \leq t \leq 20$$

Let the number of users (in thousands) on Platform B, t months after tracking begins, be modeled by the quadratic function:

$$B(t) = 5t^2 + 40t + 200, \quad 0 \leq t \leq 20$$

1. Find the number of users on both platforms at the start of the tracking period ($t = 0$). **[2 marks]**
2. Using your GDC, find the first crossover month ($t > 0$) when the user counts on both platforms are identical. **[2 marks]**
3. Using your GDC, find the second crossover month when the user counts on both platforms are identical. **[2 marks]**
4. Determine which platform has more monthly active users on Day 500 (Note: 30 days $\approx$ 1 month). Support your decision with clear calculations. **[3 marks]**

Question 14: Logarithmic Earthquake Magnitude Scale Models (Very Hard) [9 Marks]

Seismologists model the relationship between the energy released by an earthquake, E in Joules, and its Richter magnitude, M, using the logarithmic scale function:

$$M = \frac{2}{3}\log_{10}\left(\frac{E}{10^{4.8}}\right)$$

1. Show that the energy released, E, can be written explicitly in terms of M as:

$$E = 10^{1.5M+4.8}$$

[3 marks]

In a given year, two earthquakes are recorded in the South Pacific:

- Earthquake Alpha has a magnitude of $M_1 = 6.2$.
- Earthquake Beta has a magnitude of $M_2 = 8.2$.

2. Calculate the exact energy released, in Joules, by Earthquake Alpha. **[2 marks]**

3. Calculate the ratio of energy released by Earthquake Beta compared to Earthquake Alpha: $\frac{E_{\text{Beta}}}{E_{\text{Alpha}}}$. **[4 marks]**

Question 15: Logistic Disease Outbreak Population Models (Very Hard) [10 Marks]

The cumulative number of infected cases, $P(t)$, during a localized viral outbreak in an isolated community is modeled by the logistic growth function:

$$P(t) = \frac{1500}{1 + 24e^{-0.15t}}, \quad t \geq 0$$

where t is the time elapsed in days since surveillance began.

1. Find the number of infected individuals recorded on Day 0. **[1 mark]**
2. Find the horizontal asymptote of the function $y = P(t)$ as $t \to \infty$, and interpret its physical meaning in the context of the community population. **[2 marks]**
3. Find the number of days required for the cumulative infected caseload to reach exactly 1000 individuals. **[3 marks]**
4. The outbreak's daily infection rate is at its absolute peak when the caseload reaches exactly half of its long-term maximum capacity.
   (a) Determine the number of cases recorded when this peak daily rate occurs. **[1 mark]**
   (b) Calculate the specific day on which this peak occurs. **[3 marks]**

SECTION B: WORKED SOLUTIONS

Worked Solution & Mark Distribution - Question 1

1. **Company A Linear Equation:**
   Fixed cost $= 4.50$, variable cost $= 1.80$ per km:

$$F_A(x) = 1.80x + 4.50\mathbf{A1}$$

2. **Company B Linear Equation:**
   Fixed cost $= 6.00$, variable cost $= 1.50$ per km:

$$F_B(x) = 1.50x + 6.00\mathbf{A1}$$

3. **Coordinates of the Intersection Point:**
   Set the two functions equal to one another to solve for x:

$$1.80x + 4.50 = 1.50x + 6.00\mathbf{(M1)}$$

$$0.30x = 1.50 \implies x = 5 \text{ km}$$

   Substitute $x = 5$ back into either equation to calculate the fare:

$$y = 1.80(5) + 4.50 = 9.00 + 4.50 = \$13.50$$

   The coordinate intersection point is exactly $(5, 13.50)$. **A1**

4. **Physical Meaning Interpretation:**
   At exactly 5 km, both ride-share companies charge the exact same fare of \$13.50. **R1**
   For journeys shorter than 5 km, Company A is cheaper; for journeys longer than 5 km, Company B is cheaper. **R1**

Examiner Tip & Common Trap

When interpreting linear intersection points, candidates must discuss **both** variables. Do not just state that they cost the same; explicitly relate the distance traveled (5 km) to the corresponding matching cost (\$13.50).

Worked Solution & Mark Distribution - Question 2

1. **Initial Launch Height:**
   Occurs when $t = 0$:

   $$h(0) = -4.9(0)^2 + 19.6(0) + 1.5 = 1.5 \text{ m}\mathbf{A1}$$

2. **Maximum Height and Time:**
   The axis of symmetry of the parabola provides the time of maximum height:

   $$t_{\max} = \frac{-b}{2a} = \frac{-19.6}{2(-4.9)} = 2.0 \text{ seconds}\mathbf{(M1)}$$

   Substitute $t = 2.0$ back into $h(t)$ to find the peak:

   $$h(2.0) = -4.9(2)^2 + 19.6(2) + 1.5 = -19.6 + 39.2 + 1.5 = 21.1 \text{ m}$$

   The weather balloon reaches a maximum height of 21.1 **m** at exactly 2.0 **seconds**. **A1**

3. **Total Flight Duration:**
   Set $h(t) = 0$ representing the moment it returns to the ground:

   $$-4.9t^2 + 19.6t + 1.5 = 0\mathbf{(M1)}$$

   Applying the quadratic formula on GDC or manually:

   $$t = \frac{-19.6 \pm \sqrt{19.6^2 - 4(-4.9)(1.5)}}{2(-4.9)} = \frac{-19.6 \pm \sqrt{384.16 + 29.40}}{-9.8}$$

   $$t = \frac{-19.6 \pm \sqrt{413.56}}{-9.8} \implies t \approx 4.075 \text{ s (since } t \geq 0)$$

   The total flight duration is 4.08 **seconds** (to 3 s.f.). **A1**

4. **Domain and Range under Context Constraints:**
   Domain: $0 \leq t \leq 4.08$ **s**. **A1**
   Range: $0 \leq h \leq 21.1$ **m**. **A1**

Examiner Tip & Common Trap

In real-world modeling questions, do not state your domain as $-\infty < t < \infty$. Time cannot be negative, and the height model stops the instant the balloon touches the ground. Adjust boundaries to physical realities.

Worked Solution & Mark Distribution - Question 3

1. **Initial Temperature ($t = 0$):**

$$T(0) = 22 + 68e^{-0.08(0)} = 22 + 68(1) = 90°\text{C}\textbf{A1}$$

2. **Temperature after 15 minutes:**

$$T(15) = 22 + 68e^{-0.08(15)} = 22 + 68e^{-1.2}\textbf{(M1)}$$

$$T(15) = 22 + 68(0.30119...) = 22 + 20.481... = 42.481... \implies 42.5°\text{C}\textbf{A1}$$

3. **Horizontal Asymptote Analysis:**
   Equation is $T = 22$ (or $y = 22$). **A1**
   Physical Significance: 22°C represents the surrounding ambient room temperature of the environment. The mug of tea will cool down asymptotically until it reaches thermal equilibrium with the room. **R1**

4. **Cooling Time to 50°C:**
   Set $T(t) = 50$:

$$50 = 22 + 68e^{-0.08t} \implies 28 = 68e^{-0.08t}\textbf{(M1)}$$

$$e^{-0.08t} = \frac{28}{68} = \frac{7}{17} \implies -0.08t = \ln\left(\frac{7}{17}\right)$$

$$t = \frac{\ln(7/17)}{-0.08} = \frac{-0.8873}{-0.08} = 11.091... \implies 11.1\text{ minutes}\textbf{A1}$$

Examiner Tip & Common Trap

When writing horizontal asymptotes in paper 2, always write them as equations of a line (e.g., $T = 22$ or $y = 22$). Writing just "22" or "approaching 22" will lose the corresponding accuracy mark.

Worked Solution & Mark Distribution - Question 4

1. **Cost of a 1.5 kg parcel:**
   Since $1.5 \leq 2$, we use the first tier of the function:

$$C(1.5) = \$12.00\textbf{A1}$$

2. **Cost of a 5.0 kg parcel:**
   Since $2 < 5 \leq 10$, we use the second tier of the function:

$$C(5.0) = 12.00 + 2.50(5.0 - 2)\textbf{(M1)}$$

$$C(5.0) = 12.00 + 2.50(3) = 12.00 + 7.50 = \$19.50\textbf{A1}$$

3. **Weight for a cost of $\$41.00$:**
   We evaluate which tier of the function matches a cost of \$41.00. Since the maximum cost for tier 2 is $C(10) = \$32.00$, a cost of \$41.00 must belong to the third tier ($w > 10$):

$$41.00 = 32.00 + 1.80(w - 10)\textbf{(M1)}$$

$$9.00 = 1.80(w - 10) \implies w - 10 = \frac{9.00}{1.80} = 5$$

$$w = 15 \text{ kg}\textbf{A1}$$

4. **Continuity Verification at $w = 10$:**
   The left-hand limit is evaluated using tier 2:

$$\lim_{w \to 10^-} C(w) = 12.00 + 2.50(10 - 2) = 12.00 + 20.00 = 32.00$$

   The right-hand limit is evaluated using tier 3:

$$\lim_{w \to 10^+} C(w) = 32.00 + 1.80(10 - 10) = 32.00$$

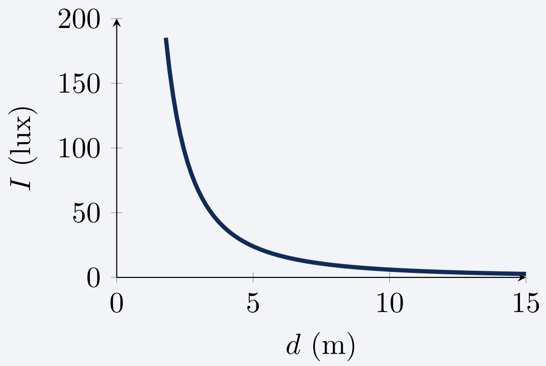
   Since the left-limit, right-limit, and function value are equal, the pricing model is **continuous** at $w = 10$. **A1**

Examiner Tip & Common Trap

For piecewise functions, continuity requires showing that evaluating the boundary value in both neighboring sections yields the exact same output. Always state both boundary substitutions explicitly on your exam script.

Worked Solution & Mark Distribution - Question 5

1. **Formulate Inverse Square Law Equation:**
   The inverse square law is written in the form:
   $$I(d) = \frac{k}{d^2}\textbf{(M1)}$$
   Substitute known parameters $d = 2$, $I = 150$:
   $$150 = \frac{k}{2^2} \implies k = 150 \times 4 = 600\textbf{(A1)}$$
   $$I(d) = \frac{600}{d^2}\textbf{A1}$$
2. **Intensity at a distance of 5 m:**
   $$I(5) = \frac{600}{5^2} = \frac{600}{25} = 24 \text{ lux}\textbf{A1}$$
3. **Distance for an intensity of 6 lux:**
   $$6 = \frac{600}{d^2} \implies d^2 = \frac{600}{6} = 100\textbf{(M1)}$$
   $$d = \sqrt{100} = 10 \text{ m (since } d > 0)\textbf{A1}$$
4. **Graph Sketch:**



Marks: Correct reciprocal-like shape in quadrant 1 showing vertical asymptotic behavior near $d \to 0$ and horizontal asymptotic behavior near the d-axis [A1].

Examiner Tip & Common Trap

When drawing reciprocal power graphs, make sure your drawn curve does not touch or cross the horizontal axis as $x \to \infty$. It must hover asymptotically above it.

Worked Solution & Mark Distribution - Question 6

1. **Show Decay Constant Equation:**
   At the half-life $t = 6.0$ hr, the mass reduces from N_0 to $0.5N_0$:

$$0.5N_0 = N_0 e^{-\lambda(6)} \implies 0.5 = e^{-6\lambda}\textbf{(M1)}$$

   Taking natural logarithms of both sides:

$$\ln(0.5) = -6\lambda \implies -\ln(2) = -6\lambda\textbf{(M1)}$$

$$\lambda = \frac{\ln(2)}{6}\textbf{A1AG}$$

2. **Remaining Isotope Mass after 15 hours:**
   Using GDC to evaluate the unrounded exponent base:

$$N(15) = 200e^{-\left(\frac{\ln(2)}{6}\right)\times 15} = 200e^{-2.5\ln(2)}\textbf{(M1)}$$

$$N(15) = 200 \times 2^{-2.5} = 200 \times 0.17677... = 35.3553... \implies 35.4 \text{ mg}\textbf{A1}$$

3. **Decay Time to 15 mg:**
   Set $N(t) = 15$:

$$15 = 200e^{-\lambda t} \implies e^{-\lambda t} = \frac{15}{200} = 0.075\textbf{(M1)}$$

   Take logs to isolate t:

$$-\lambda t = \ln(0.075) \implies t = \frac{\ln(0.075)}{-\frac{\ln(2)}{6}}\textbf{(M1)}$$

$$t = \frac{-2.59026...}{-0.11552...} = 22.421... \implies 22.4 \text{ hours}\textbf{A1}$$

4. **Mass Decayed Away over 24 hours:**
   Notice that 24 hours is exactly four half-lives ($24/6 = 4$ half-lives):

$$\text{Mass Remaining} = 200 \times \left(\frac{1}{2}\right)^4 = 200 \times \frac{1}{16} = 12.5 \text{ mg}$$

$$\text{Decayed Mass} = 200 - 12.5 = 187.5 \text{ mg} \implies 188 \text{ mg}\textbf{A1}$$

Examiner Tip & Common Trap

Be careful with the distinction between the "remaining mass" (found directly from $N(t)$) and the "decayed mass". The decayed mass is calculated by taking the initial mass and subtracting the remaining mass: $N_0 - N(t)$.

Worked Solution & Mark Distribution - Question 7

1. **Initial Reservoir Volume ($t = 0$):**

$$V(0) = -0^3 + 12(0)^2 - 36(0) + 200 = 200 \text{ dam}^3\textbf{A1}$$

2. **Find Derivative $V'(t)$:**
Using the power rule term-by-term:

$$V'(t) = -3t^2 + 24t - 36\textbf{A1A1}$$

3. **Turning Points Algebra (Setting $V'(t) = 0$):**

$$-3t^2 + 24t - 36 = 0 \implies t^2 - 8t + 12 = 0\textbf{(M1)}$$

Factorising the quadratic:

$$(t - 2)(t - 6) = 0 \implies t = 2 \text{ and } t = 6\textbf{(A1)}$$

Calculate corresponding V-coordinates:

$$V(2) = -(2)^3 + 12(2)^2 - 36(2) + 200 = -8 + 48 - 72 + 200 = 168 \text{ dam}^3$$

$$V(6) = -(6)^3 + 12(6)^2 - 36(6) + 200 = -216 + 432 - 216 + 200 = 200 \text{ dam}^3$$

The turning points are **Local Min at** $(2, 168)$ and **Local Max at** $(6, 200)$. **A1**

4. **Increasing Water Interval:**
The function increases strictly between the local minimum and local maximum:

$$2 < t < 6 \text{ weeks (or)}\textbf{A1}$$

5. **Absolute Minimum over Closed Domain $[0, 10]$:**
Test values at the boundaries of the domain alongside local extrema:

$$V(0) = 200, \quad V(2) = 168, \quad V(6) = 200$$

$$V(10) = -(10)^3 + 12(10)^2 - 36(10) + 200 = -1000 + 1200 - 360 + 200 = 40 \text{ dam}^3\textbf{(M1)}$$

The absolute minimum volume occurs at the end of the cycle: 40 **dam**^{3}. **A1**

Examiner Tip & Common Trap

When asked for an absolute maximum or minimum over a closed interval $[a, b]$, you must evaluate the function values at the endpoints a and b as well as at any turning points. Here, the end volume on Week 10 (40) is much lower than the local minimum on Week 2 (168).

Worked Solution & Mark Distribution - Question 8

1. **Amplitude a and Vertical Shift d:**
   Minimum height $= 5$, Maximum height $= 80 + 5 = 85$ m.

$$a = \frac{\text{Max} - \text{Min}}{2} = \frac{85 - 5}{2} = 40\textbf{A1}$$

$$d = \frac{\text{Max} + \text{Min}}{2} = \frac{85 + 5}{2} = 45\textbf{A1}$$

2. **Find Frequency Coefficient b:**
   The period of the rotation is 12 minutes:

$$\text{Period} = \frac{2\pi}{b} \implies 12 = \frac{2\pi}{b}\textbf{(M1)}$$

$$b = \frac{2\pi}{12} = \frac{\pi}{6}\textbf{A1}$$

3. **Cabin Height after 4 minutes:**
   Ensure your GDC is strictly set to **Radian Mode**:

$$H(4) = -40\cos\left(\frac{\pi}{6} \times 4\right) + 45 = -40\cos\left(\frac{2\pi}{3}\right) + 45\textbf{(M1)}$$

$$H(4) = -40(-0.5) + 45 = 20 + 45 = 65 \text{ m}\textbf{A1}$$

4. **Time Interval Above 65 m:**
   Set $H(t) > 65$:

$$-40\cos\left(\frac{\pi}{6}t\right) + 45 = 65 \implies \cos\left(\frac{\pi}{6}t\right) = -0.5\textbf{(M1)}$$

   For the first cycle ($0 \leq t \leq 12$):

$$\frac{\pi}{6}t = \frac{2\pi}{3} \implies t = 4 \text{ minutes}$$

$$\frac{\pi}{6}t = \frac{4\pi}{3} \implies t = 8 \text{ minutes}$$

   The duration spent above 65 m is: $8 - 4 = 4$ minutes. **A1**

Examiner Tip & Common Trap

Failing to switch the GDC angle setting to Radians is the single most common error on trigonometric modeling problems. If the variable is time t, and the coefficient b involves π, the GDC must be set to Radians.

Worked Solution & Mark Distribution - Question 9

1. **Average Cost Function $A(x)$:**

$$A(x) = \frac{1500 + 3.5x}{x} = \frac{1500}{x} + 3.5\textbf{A1}$$

2. **Vertical Asymptote:**
   Equation is $x = 0$. **A1**
   Economic Meaning: As the printing volume approaches 0, the fixed setup cost of $1500 is spread over almost zero pages, making the average cost per page skyrocket towards infinity. **R1**

3. **Horizontal Asymptote:**
   As $x \to \infty$, the term $\frac{1500}{x} \to 0$:

$$\lim_{x\to\infty} A(x) = 3.5 \implies y = 3.5\textbf{A1}$$

   Economic Meaning: $3.50 per thousand pages is the limiting base marginal printing cost as fixed costs become negligible due to extreme high-volume scaling. **R1**

4. **Crossover Calculation:**
   Find when the average cost $A(x)$ is strictly cheaper than the flat fee rate $S(x)$:

$$\frac{1500}{x} + 3.5 < 4.2\textbf{(M1)}$$

$$\frac{1500}{x} < 0.7 \implies x > \frac{1500}{0.7}\textbf{(M1)}$$

$$x > 2142.857\ldots \text{ thousand pages}$$

   The firm must print at least 2143 **thousand pages** daily. **A1**

Examiner Tip & Common Trap

When solving real-world inequalities, verify rounding direction. To be *strictly cheaper*, you must pass the exact real root. Producing 2142 thousand pages leaves you slightly more expensive; 2143 thousand pages is the first integer crossing the threshold.

Worked Solution & Mark Distribution - Question 10

1. **Constructing Equations from Points:**
   Substitute $P_1(2, 10)$, $P_2(4, 30)$, and $P_3(6, 62)$ into $y = ax^2 + bx + c$:

$$a(2)^2 + b(2) + c = 10 \implies 4a + 2b + c = 10\textbf{A1}$$
$$a(4)^2 + b(4) + c = 30 \implies 16a + 4b + c = 30\textbf{A1}$$
$$a(6)^2 + b(6) + c = 62 \implies 36a + 6b + c = 62\textbf{A1}$$

2. **Matrix Formulation:**

$$\begin{pmatrix} 4 & 2 & 1 \\ 16 & 4 & 1 \\ 36 & 6 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 10 \\ 30 \\ 62 \end{pmatrix} \textbf{A2}$$

   Note: Award A2 for full matrix form; A1 if matrices are correct but vectors are misplaced.

3. **Solving with GDC:**
   Entering the system into GDC matrix calculation $[A]^{-1}[B]$ or Simultaneous Solver:

$$a = 1.5, \quad b = 1.0, \quad c = 2.0\textbf{A2}$$

   Note: Award A1 for two correct values, A2 for all three correct values.

4. **Height of Cable at $x = 8$:**
   Using $y = 1.5x^2 + x + 2$:

$$y(8) = 1.5(8)^2 + 8 + 2 = 1.5(64) + 10 = 96 + 10 = 106 \text{ units}\textbf{A1}$$

Examiner Tip & Common Trap

When constructing systems of equations to find parabola coefficients, ensure you square the x-values correctly. Many students make the mistake of writing $2a+2b+c$ instead of $4a+2b+c$.

Worked Solution & Mark Distribution - Question 11

1. **Show Linearisation with Logarithms:**
Take natural logarithms of both sides of $E = aM^b$:

$$\ln(E) = \ln(aM^b)\mathbf{M1}$$

Using properties of logarithms ($\ln(xy) = \ln(x) + \ln(y)$ and $\ln(x^y) = y\ln(x)$):

$$\ln(E) = \ln(a) + \ln(M^b) \implies \ln(E) = b\ln(M) + \ln(a)\mathbf{A1AG}$$

2. **Calculate Slope Parameter b:**
Using coordinate points $(X_1, Y_1) = (2.77, 6.33)$ and $(X_2, Y_2) = (4.39, 7.54)$:

$$b = \frac{\Delta Y}{\Delta X} = \frac{7.54 - 6.33}{4.39 - 2.77}\mathbf{(M1)(A1)}$$

$$b = \frac{1.21}{1.62} = 0.746913... \implies b \approx 0.75\mathbf{A1}$$

3. **Calculate Scale Constant a:**
Substitute $b = 0.746913$ into either coordinates to find the vertical intercept:

$$6.33 = 0.746913(2.77) + \ln(a)\mathbf{(M1)}$$

$$6.33 = 2.06895... + \ln(a) \implies \ln(a) = 4.26105...$$

Convert out of logs:

$$a = e^{4.26105...} = 70.884... \implies a \approx 71\mathbf{A1}$$

4. **Estimated Metabolism for a 16 kg Mammal:**
Using $E = 71M^{0.75}$:

$$E = 71(16)^{0.75} = 71 \times 8 = 568 \text{ kcal/day}\mathbf{(M1)A1}$$

Note: Accept answers around 560 kcal/day if candidates rounded $a = 70$ using a different intermediate rounding pathway.

Examiner Tip & Common Trap

Logarithmic transformations are a popular IB topic. Make sure to clearly state your variables: here, the linear equation $Y = mX + c$ equates to $Y = \ln(E)$, $X = \ln(M)$, $m = b$, and $c = \ln(a)$. Always exponentiate e^c to get the final scale constant a.

Worked Solution & Mark Distribution - Question 12

1. **Continuous Boundary Constraint:**
   For continuity at $x = 60$, the cost evaluated from the left must equal the cost evaluated from the right:

$$C(60) = -0.05(60)^2 + 8.00(60) + 50.00 = m(60) + c\textbf{(M1)}$$

$$C(60) = -180.00 + 480.00 + 50.00 = 350.00$$

   Therefore:

$$60m + c = 350\textbf{A1}$$

2. **Show Differentiable Slope Requirement:**
   Find the derivative of the first part of the piecewise function:

$$C_1'(x) = -0.1x + 8.00\textbf{A1}$$

   Evaluate the derivative at the boundary $x = 60$:

$$C_1'(60) = -0.1(60) + 8.00 = -6.00 + 8.00 = 2.00\textbf{A1}$$

   The derivative of the second part of the piecewise function is simply:

$$C_2'(x) = m$$

   To prevent any sharp kinks, the slopes must match:

$$m = 2.00\textbf{A1AG}$$

3. **Find Constant c:**
   Substitute $m = 2.00$ into the continuity equation from part (1):

$$60(2.00) + c = 350 \implies 120 + c = 350\textbf{(M1)}$$

$$c = 230\textbf{A1}$$

4. **Total Cost of 100 kg Output:**
   Since $100 > 60$, we use the linear portion of our piecewise function:

$$C(100) = 2.00(100) + 230\textbf{(M1)}$$

$$C(100) = 200 + 230 = \$430.00\textbf{A1}$$

Examiner Tip & Common Trap

Differentiability of piecewise models is an advanced modeling topic. It simply means the slopes of the neighboring curves match up perfectly at the transition point. Set the derivative of the left section equal to the derivative of the right section at the boundary.

Worked Solution & Mark Distribution - Question 13

1. **Initial User Counts ($t = 0$):**

$$A(0) = 250(1.15)^0 = 250 \text{ thousand}\mathbf{A1}$$
$$B(0) = 5(0)^2 + 40(0) + 200 = 200 \text{ thousand}\mathbf{A1}$$

2. **First Crossover Month:**
We want to solve the intersection equation $A(t) = B(t)$:

$$250(1.15)^t = 5t^2 + 40t + 200\mathbf{(M1)}$$

Enter functions into GDC graphing mode and search for intersections:

$$Y_1 = 250(1.15)^x, \quad Y_2 = 5x^2 + 40x + 200$$

The first intersection occurs at:

$$t = 3.853... \implies 3.85 \text{ months}\mathbf{A1}$$

3. **Second Crossover Month:**
Using the GDC graph intersection tool further down the timeline:

$$t = 13.530... \implies 13.5 \text{ months}\mathbf{A1}$$

4. **platform comparison on Day 500:**
First, convert 500 days into timeline months:

$$t = \frac{500}{30} = 16.666... \text{ months}\mathbf{(M1)}$$

Substitute $t = 16.67$ into both models:

$$A(16.666) = 250(1.15)^{16.666...} \approx 2555.51 \text{ thousand}\mathbf{(A1)}$$

$$B(16.666) = 5(16.666...)^2 + 40(16.666...) + 200 \approx 2255.56 \text{ thousand}\mathbf{(A1)}$$

Therefore, **Platform A** has more active users (approx. 2.56 million compared to Platform B's 2.26 million). **A1**

Examiner Tip & Common Trap

When using numerical solvers or GDC graph screens to find crossover intersections, expand your visual window to capture all possible intersection points. For exponential vs. polynomial models, there are often two intersections.

Worked Solution & Mark Distribution - Question 14

1. **Show Energy Equation Manipulation:**

$$M = \frac{2}{3}\log_{10}\left(\frac{E}{10^{4.8}}\right)$$

Multiply both sides by $\frac{3}{2}$:

$$1.5M = \log_{10}\left(\frac{E}{10^{4.8}}\right) \textbf{M1}$$

Convert from logarithmic form to exponential form:

$$10^{1.5M} = \frac{E}{10^{4.8}}\textbf{M1}$$

Multiply by $10^{4.8}$ to isolate E:

$$E = 10^{1.5M} \times 10^{4.8} = 10^{1.5M+4.8}\textbf{A1AG}$$

2. **Energy Released by Earthquake Alpha:**
Substitute $M_1 = 6.2$:

$$E_{\text{Alpha}} = 10^{1.5(6.2)+4.8} = 10^{9.3+4.8}\textbf{(M1)}$$

$$E_{\text{Alpha}} = 10^{14.1} = 1.25892... \times 10^{14} \text{ Joules} \implies 1.26 \times 10^{14} \text{ Joules}\textbf{A1}$$

3. **Energy Release Ratio:**
Substitute $M_2 = 8.2$:

$$E_{\text{Beta}} = 10^{1.5(8.2)+4.8} = 10^{12.3+4.8} = 10^{17.1} \text{ Joules}\textbf{(A1)}$$

Set up ratio:

$$\text{Ratio} = \frac{10^{17.1}}{10^{14.1}}\textbf{(M1)}$$

Using laws of indices ($10^a/10^b = 10^{a-b}$):

$$\text{Ratio} = 10^{17.1-14.1} = 10^{3.0}\textbf{(A1)}$$

Earthquake Beta released exactly 1000 **times** more energy than Earthquake Alpha. **A1**

Examiner Tip & Common Trap

When calculating ratios of large exponential outputs, divide the base expressions rather than dividing the calculated decimals. $10^{17.1}/10^{14.1}$ simplifies cleanly to 10^3, which is exactly 1000 without any rounding error.

Worked Solution & Mark Distribution - Question 15

1. **Infected Caseload on Day 0:**

$$P(0) = \frac{1500}{1 + 24e^{-0.15(0)}} = \frac{1500}{1 + 24(1)} = \frac{1500}{25} = 60 \text{ cases}\mathbf{A1}$$

2. **Horizontal Asymptote Analysis:**
As $t \to \infty$, the term $e^{-0.15t} \to 0$:

$$\lim_{t\to\infty} P(t) = \frac{1500}{1 + 24(0)} = 1500 \implies y = 1500\mathbf{A1}$$

Physical Meaning: 1500 represents the carrying capacity of the community. It represents the ultimate maximum limit of total people who can be infected. **R1**

3. **Time to Reach 1000 Cases:**
Set $P(t) = 1000$:

$$1000 = \frac{1500}{1 + 24e^{-0.15t}} \implies 1 + 24e^{-0.15t} = \frac{1500}{1000} = 1.5\mathbf{(M1)}$$

$$24e^{-0.15t} = 0.5 \implies e^{-0.15t} = \frac{0.5}{24} = \frac{1}{48}\mathbf{(M1)}$$

$$-0.15t = \ln\left(\frac{1}{48}\right) \implies t = \frac{-\ln(48)}{-0.15} = 25.808... \implies 25.8 \text{ days}\mathbf{A1}$$

4. **Outbreak Infection Rate Peak Analysis:**

   (a) Peak occurs at half the carrying capacity:

$$P_{\text{peak}} = \frac{1500}{2} = 750 \text{ cases}\mathbf{A1}$$

   (b) Set $P(t) = 750$:

$$750 = \frac{1500}{1 + 24e^{-0.15t}} \implies 1 + 24e^{-0.15t} = 2\mathbf{(M1)}$$

$$24e^{-0.15t} = 1 \implies e^{-0.15t} = \frac{1}{24}\mathbf{(M1)}$$

$$-0.15t = \ln\left(\frac{1}{24}\right) \implies t = \frac{-\ln(24)}{-0.15} = 21.186... \implies 21.2 \text{ days}\mathbf{A1}$$

Examiner Tip & Common Trap

The point on a logistic curve where the rate of change is at its peak is mathematically the inflection point. You do not need complex derivatives to find it: it is always located at exactly half of the horizontal carrying capacity ($y = L/2$).